

SPECTRAL ANALYSIS AND DIRICHLET FORMS ON BARLOW-EVANS FRACTALS

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ABSTRACT. We show that if a Barlow-Evans Markov process on a vermiculated space is symmetric, then one can study the spectral properties of the corresponding Laplacian using projective limits. For some examples, such as the Laakso spaces and a Sierpinski Pâte à Choux, one can develop a complete spectral theory, including the eigenfunction expansions that are analogous to Fourier series. In addition, we construct connected fractal spaces isospectral to the fractal strings of Lapidus and van Frankenhuysen.

1. INTRODUCTION

We study symmetric regular Dirichlet forms [12] on the fractal-like spaces F_∞ constructed in [10]. Barlow and Evans in [10] describe a construction for a new, interesting class of state spaces for Markov processes utilizing projective limits. We furthermore assume that the base spaces are Dirichlet metric measure spaces, that is, metric measure spaces equipped with Dirichlet forms. We show that in this case one can develop a complete spectral theory of the associated Laplace operators, including formulas for spectral projections, utilizing the tools of Dirichlet form theory. The characterization of the spectra of the Laplacians presented here is a generalization of those obtained previously by the first author for Laakso spaces in [21, 23].

Given a measure space on which one has a Laplacian it is natural to study the spectrum. As the measure space becomes more complicated this task can become very difficult. On fractal spaces such as the Sierpinski gasket and carpet this problem has been extensively studied [25, 7, 8, 16, 17]. For finitely ramified self-similar highly symmetric fractals a complete spectral analysis is possible although rather complicated, see [2, 3] and references therein. Moreover, it is possible to extend this kind of spectral analysis to finitely ramified fractafolds, that is to metric measure spaces that have local charts from open sets of a reference fractal. This is one way of obtaining new examples from old, including isospectral fractafolds, see [25, 26]. The projective limit construction provides yet another way of controllably obtaining new measure spaces and in this paper we examine how the spectral data transfers to the limit.

The main goal of this paper is an understanding of the spectrum of a class of Laplacians. We have found that working in terms of the associated Dirichlet form to be more straight forward. This is particularly noticeable in Definition 2.4. Where the domain of a Dirichlet form is much easier to write down than the domain of the corresponding Laplacian.

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As we discuss in the final section of this paper, the projective limit construction can produce many connected fractals which are isospectral to a given fractal string (see [20] and references therein). Which makes it possible to ask which fractal strings then correspond to connected fractals on which there are heat kernel estimates for a diffusion process. However, answering this question is beyond the scope of this paper. Determining heat kernel estimates on fractal spaces has a long tradition (see for instance, [4, 5, 6, 9, 24] and an extensive review in [13]). For example Laakso spaces have Gaussian heat kernel estimates while Sierpinski gasket-like fractals have sub-Gaussian estimates often depending on geometric conditions.

One note of caution, our analysis of fractals defined as projective limits is entirely intrinsic on abstractly defined objects. Even in the simplest examples, Laakso spaces, the limit space is not bi-Lipschitz embeddable in any finite dimensional Euclidean space.

We begin with definitions in Section 2. In Sections 3 and 4 we provide the background on projective systems of measure spaces along with the limiting procedure for the Laplacians on each approximating measure space. Section 5 contains the main results of the paper which give a decomposition of the spectrum of the Laplacian on the limit space. Then in Section 6 we describe three classes of examples of spaces that can be constructed with this method, and outline the computation of the spectrum in each case.

2. DEFINITIONS

The following definitions are based on those given in [10] and references therein.

Definition 2.1. *Let F_0 be a locally compact second-countable Hausdorff space with a σ -finite Borel measure μ_{F_0} . In addition we assume there is a sequence of compact second-countable Hausdorff spaces G_i for $i \geq 0$ with Borel probability measures μ_{G_i} .*

Inductively define a sequence of locally compact topological measure spaces and maps between them as follows (refer to Figure 1). Suppose $i > 0$, F_{i-1} is defined, (is a locally compact second-countable Hausdorff space,) and $B_i \subset F_{i-1}$ is closed. Set

$$F_i = (F_{i-1} \setminus B_i) \times G_i \cup B_i$$

and

$$\pi_i(x, g) = \begin{cases} (x, g) & \text{if } x \in F_{i-1} \setminus B_{i-1} \\ x & \text{if } x \in B_{i-1}. \end{cases}$$

The space F_i is topologized by the map π_i , which means that a subset of F_i is open if and only if its π_i preimage is open.

The map ψ_i is the natural projection from $F_{i-1} \times G_i \mapsto F_{i-1}$ and define $\phi_i = \psi_i \circ \pi_i^{-1} : F_i \mapsto F_{i-1}$. Alternatively ϕ_i can be defined by

$$\begin{aligned} \phi_i(x, g) &= x & \text{if } x \in F_{i-1} \setminus B_{i-1} \\ \phi_i(x) &= x & \text{if } x \in B_{i-1}. \end{aligned}$$

Definition 2.2. *The sequence of spaces and associated maps $\{F_i, G_i, \phi_i, \pi_i, \psi_i\}$ will be called a Barlow-Evans sequence.*

For the rest of the paper a space F_i will be assumed to be a member of a Barlow-Evans sequence with all the associated maps assumed. For any $M = 0, 1, \dots, \infty$ we shall denote the L^2 norm on functions over F_i by $\|\cdot\|_M$. These norms should not be confused for the L^p norms which are not used in this paper.

If f is a function on F_i then $\pi_i^* f$ is a function on $F_{i-1} \times G_i$ defined by

$$\pi_i^* f = f \circ \pi_i.$$

Similarly for ψ_i^* , ϕ_i^* and also set $\phi_{ij}^* = \phi_i^* \circ \dots \circ \phi_{j+1}^*$ taking functions on F_j to functions on F_i .

Proposition 2.1. *For all $i \geq 1$*

$$f \in C_0[F_i] \text{ if and only if } \pi_i^* f \in C_0[F_{i-1} \times G_i].$$

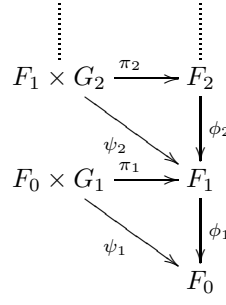


FIGURE 1. The sequence of spaces and the maps between them in Definition 2.1.

$$\begin{array}{ccccccc}
C_0(F_{i-1} \times G_i) & \hookrightarrow & L_{i-1}^2 \otimes L_{G_i}^2 & \hookleftarrow & \mathcal{F}_{i-1} \otimes L_{G_i}^2 & \hookleftarrow & \text{Dom}(\Delta_{i-1}) \otimes L_{G_i}^2 \\
\uparrow \pi_i^* & & \uparrow \pi_i^* & & \uparrow \pi_i^* & & \uparrow \pi_i^* \\
C_0(F_i) & \hookrightarrow & L_i^2 & \hookleftarrow & \mathcal{F}_i & \hookleftarrow & \text{Dom}(\Delta_i)
\end{array}$$

FIGURE 2. A diagram of function spaces. The vertical injections are isometric and induced by the map π_i .

Proof. Recall that F_i has the induced topology from $F_{i-1} \times G_i$ by π_i . That is $U \subset F_i$ is open if and only if $\pi_i^{-1}(U) \subset F_{i-1} \times G_i$ is open. Let $V \subset \mathbb{R}$, then $f^{-1}(V) \subset F_i$ is open if and only if $\pi_i^{-1}(f^{-1}(V)) \subset F_{i-1} \times G_i$ is open. But $\pi_i^{-1}(f^{-1}(V)) = (\pi_i^* f)^{-1}(V)$ so as V ranges over all open sets of \mathbb{R} we have that f and $\pi_i^* f$ are both either continuous or discontinuous. \square

Definition 2.3. Given μ_{F_0} inductively define measures μ_{F_i} on F_i for $i \geq 1$ by

$$\int_{F_i} f \, d\mu_{F_i} = \int_{F_{i-1} \times G_i} \pi_i^* f \, d(\mu_{F_{i-1}} \times \mu_{G_i}).$$

The measure μ_{F_i} is defined on the Borel σ -algebra generated by the above defined topology on F_i .

Note that if μ_{F_0} is a finite measure with mass $|\mu_{F_0}|$ then all μ_{F_i} have the same total mass. Thus we have $L^2(F_i, \mu_i)$ for all i on which we define Dirichlet forms.

Definition 2.4. If a Dirichlet form $(\mathcal{E}_{i-1}, \mathcal{F}_{i-1})$ on F_{i-1} is regular then define $(\mathcal{E}_i, \mathcal{F}_i)$ by

$$\mathcal{E}_i(f, f) = \int_{G_i} \mathcal{E}_{i-1}(\pi_i^* f, \pi_i^* f) \, d\mu_{G_i}$$

and \mathcal{F}_i to be the closure in the $\|\cdot\|_i^2 + \mathcal{E}_i(\cdot, \cdot)$ metric of $\mathfrak{F}_i = \{f \in L^2(F_i, \mu_{F_i}) \mid \pi_i^* f(x, g) = \sum_{k=1}^n f_k(x) h_k(g), f_k \in \mathcal{F}_{i-1}, h_k \in C(G_i)\}$.

Note that $\pi_i^* f$ is a function in two variables, one along F_{i-1} and another along G_i . So this definition can be read as applying \mathcal{E}_{i-1} to $\pi_i^* f$ for each element of G_i and then integrating over G_i . Before examining the properties of $(\mathcal{E}_i, \mathcal{F}_i)$ we make sure it is well defined.

Lemma 2.1. For $f \in \mathfrak{F}_i$ $\mathcal{E}_{i-1}(\pi_i^* f, \pi_i^* f)(g)$ is μ_{G_i} -measurable.

Proof. Let $\pi_i^* f = f_1(x) h_1(g)$ where $f_1 \in \mathcal{E}_{i-1}$ and $h_1 \in C(G_i)$. Then

$$\mathcal{E}_{i-1}(\pi_i^* f, \pi_i^* f)(g) = \mathcal{E}_{i-1}(f_1(x) h_1(g), f_1(x) h_1(g)) = h_1^2(g) \mathcal{E}_{i-1}(f_1, f_1)$$

Which is a continuous function of $g \in G_i$ and hence Borel measurable. This extends to finite linear combinations naturally. \square

Lemma 2.2. If \mathcal{E}_{i-1} is regular then for $i \geq 1$ $\mathfrak{F}_i \cap C_0[F_i]$ is a dense subset of $C_0[F_i]$.

Proof. Fix $f \in C_0[F_i]$ and $\epsilon > 0$. Let $X = \pi_i(\psi_i^{-1} \circ \psi_i(\text{supp}[\pi_i^* f])) \subset F_i$. Then X is compact and $f = 0$ on ∂X . Then on X , the collection of elements of \mathfrak{F}_i restricted to X is closed under products and contains constant functions and separates points of F_i . By the Stone-Weierstrass Theorem \mathfrak{F}_i restricted to X is dense in $C[X]$ so there exists a finite combination of elements of \mathfrak{F}_i that approximate f uniformly within an error of ϵ on X such that the supports of these functions are contained in the support of f . This finite sum of elements of \mathfrak{F}_i restricted to X all vanish in a neighborhood of ∂X so as functions they can be continued to all of F_i by setting them equal to zero on X^C . \square

Theorem 2.1. *If \mathcal{E}_0 is a regular Dirichlet form, then \mathcal{E}_i are also regular Dirichlet forms for all $i \geq 0$. Moreover, if \mathcal{E}_0 is local then \mathcal{E}_i is local as well.*

Proof. We proceed by induction. The base case is given as an assumption so it suffices to show that if \mathcal{E}_{i-1} is a local regular Dirichlet form then so is \mathcal{E}_i . By Lemma 2.1 \mathcal{E}_i is well defined. The locality, linearity, non-negativity, and Markovian property of \mathcal{E}_i follow directly from the definition and the fact that μ_{G_i} is a positive probability measure.

The form \mathcal{E}_i must be shown to be closed for it to be a Dirichlet form. To be closed \mathcal{F}_i must be not only closed but complete under the $\|\cdot\|_i^2 + \mathcal{E}_i(\cdot, \cdot)$ metric. Assume that $u_n \in \mathfrak{F}_i$ be Cauchy and that $\|u_n\|_i \rightarrow 0$. This is sufficient since \mathcal{F}_i is the closure of this set. If u_n are Cauchy in the $\|\cdot\|_i + \mathcal{E}(\cdot, \cdot)$ metric then it is Cauchy in $\|\cdot\|_i$. By Definition 2.3 this can be interpreted as $\pi_i^* u_n(x, g)$ is L^2 convergent in the x -variable thus there is an almost everywhere convergent subsequence $u_{n_k} = u_k$ so that $\|\pi_i^* u_k(\cdot, g)\|_{i-1} \rightarrow 0$ for almost every $g \in G_i$. It remains to show that $f_p(g) = \mathcal{E}_{i-1}(\pi_i^* u_p(\cdot, g))$ is for almost all g Cauchy for some subsequence of $u_{k_p} = u_p$.

In order to see that $f_p(g)$ is Cauchy for almost every g for some subsequence $u_{k_p} = u_p$ recall that u_n is Cauchy in the $\|\cdot\|_i + \mathcal{E}(\cdot, \cdot)$ metric so $\mathcal{E}_i(u_n, u_n)$ is a Cauchy sequence of positive numbers. In Definition 2.4 the form \mathcal{E}_i is defined as the integral over G_i of $f_n(g)$ so is a $L^1(G_i)$ convergence sequence. This means that along the subsequence u_p $f_p(g)$ converges for almost every g . Since $\pi_i^* u_p(\cdot, g) \rightarrow 0$ then $f_p(g) \rightarrow 0$ by the fact that \mathcal{E}_{i-1} is assumed to be closed by the induction hypothesis. Which is by [12] sufficient for \mathcal{E}_i to be a closed form on \mathcal{F}_i .

By Lemma 2.2 $\mathfrak{F}_i \cap C_0(F_i)$ is uniformly dense in $C_0(F_i)$. By construction of $(\mathcal{E}_i, \mathcal{F}_i)$ \mathfrak{F}_i is a dense subset of \mathcal{F}_i in the $\|\cdot\|_i^2 + \mathcal{E}_i(\cdot, \cdot)$ metric. Hence \mathcal{E}_i is regular if \mathcal{E}_{i-1} is regular.

Let $u, v \in \mathcal{E}_i$ have disjoint support. Then $\pi_i^* u_i$ and $\pi_i^* v$ have disjoint support in $F_{i-1} \times G_i$. Since \mathcal{E}_{i-1} is by assumption local then for each $g \in G_i$ $\pi_i^* u(\cdot, g)$ and $\pi_i^* v(\cdot, g)$ have disjoint support and thus $\mathcal{E}_{i-1}(\pi_i^* u, \pi_i^* v)(g) = 0$ for all $g \in G_i$. Then taking the integral over G_i we get that $\mathcal{E}_i(u, v) = 0$. Thus if \mathcal{E}_{i-1} is local then \mathcal{E}_i is local. \square

Corollary 2.1. *The domains of the Dirichlet forms \mathcal{E}_i are nested. That is*

$$\phi_i^* \mathcal{F}_{i-1} \subset \mathcal{F}_i.$$

Proposition 2.2. *If $f \in \mathcal{F}_i$ then $\pi_i^* f(\cdot, g) \in \mathcal{F}_{i-1}$ for almost every $g \in G_i$ and $\mathcal{E}_{i-1}(\pi_i^* f)(g) \in L^1(G_i)$.*

Proof. This holds if

$$f \in \{f \in L^2(F_i, \mu_{F_i}) | \pi_i^* f(x, g) = \sum_{k=1}^n f_k(x) h_k(g),$$

Where $f_k \in \mathcal{F}_{i-1}$, $h_k \in C(G_i)\}$. Thus it only remains to show that these properties are preserved under the $\|\cdot\|_i^2 + \mathcal{E}_i(\cdot, \cdot)$ norm. The continuation of these properties follow from the proof of Theorem 2.1, in particular the proof of the closability of \mathcal{E}_i . \square

3. PROJECTIVE LIMITS

The construction that is considered in this paper is a means of constructing state spaces for the symmetric diffusions via projective limit. Effectively this process takes compatible sequences of topological spaces and taking their limit. Barlow and Evans [10] considered this construction as a way to produce exotic state spaces for Markov processes. Then [21] specialized Barlow and Evans work to Laakso spaces [18].

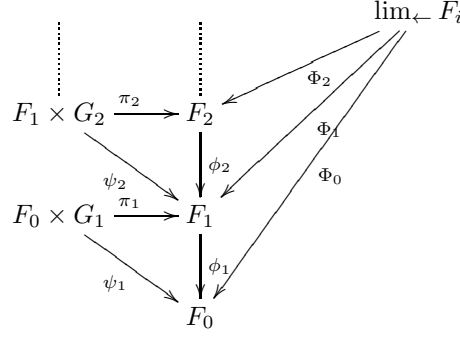


FIGURE 3. The projective system of spaces and the maps between them.

Definition 3.1. A projective system of measure spaces is a collection of measure spaces (F_i, μ_{F_i}) with projections $\phi_i : F_i \rightarrow F_{i-1}$ such that for a $\mu_{F_{i-1}}$ -measurable set A that $\mu_{F_i}(\phi_i^{-1}(A)) = \phi_i^* \mu_{F_i}(A) = \mu_{F_{i-1}}(A)$. Then a projective limit can be defined as $\lim_{\leftarrow} F_i \subset \prod_i F_i$ as the set of sequences $x_i \in \prod_i F_i$ that have the property $\phi_i(x_i) = x_{i-1}$. Then the canonical projections $\Phi_j : \prod_i F_i \rightarrow F_j$ can be restricted to $\lim_{\leftarrow} F_i$ and have the consistency property:

$$\phi_i \circ \Phi_i = \phi_{i-1}, \quad i \geq 1.$$

Proposition 3.1 ([14]). There exists a unique measure on $\lim_{\leftarrow} F_i$ denoted μ_{F_∞} if the masses of μ_{F_i} are uniformly bounded. Then μ_{F_∞} satisfies

$$(1) \quad \mu_{F_i}(A) = \mu_{F_\infty}(\Phi_i^{-1}(A))$$

for all A that are μ_{F_i} -measurable.

Corollary 3.1. If μ_{F_0} is σ -finite then there exists a unique μ_{F_∞} on F_∞ which satisfies (1).

Proof. Since μ_{F_i} is σ -finite there exists a partition of F_0 such that each element of the partition has finite measure. Apply Proposition 3.1 on each member of the partition and then take μ_{F_∞} to be their sum. \square

We shall often have probability measures on F_i so that it will be possible to consider directly the limit measure space $(\lim_{\leftarrow} F_i, \mu_{F_\infty})$ without the worry of this corollary. Note that as maps from the Borel functions on F_i to the Borel functions on F_∞ that Φ_i^* are \mathbb{R} -linear maps.

Proposition 3.2. Let $\text{clos}_{\text{uniform}}$ represent the closure operator in the uniform norm then

$$C_0(F_\infty) = \text{clos}_{\text{uniform}} \left\{ \bigcup_{i=0}^{\infty} \Phi_i^* C_0(F_i) \right\}.$$

4. PROJECTIONS AND LAPLACIANS

Having constructed Dirichlet forms on the approximating F_i we now turn to a Dirichlet form over F_∞ . Recall that the $L^2(F_M, \mu_{F_M})$ norm by $\|\cdot\|_M$ for $M = 0, 1, 2, \dots, \infty$. The existence of projective limits of Dirichlet spaces (L^2 spaces equipped with a Dirichlet form and its domain) is briefly discussed in [11]. We develop the existence for the sake of the accompanying notation which is then used to describe the decompositions in Theorem 4.3. The decompositions rely on the specific structure of the equivalence relations used in defining the projective system and are not a general feature of projective systems of Dirichlet spaces.

Definition 4.1. Given a Barlow-Evans sequence let \mathcal{E}_∞ be the quadratic form on $\lim_{\leftarrow} F_i$ defined by

$$\mathcal{E}_\infty(\Phi_i^* u, \Phi_i^* u) = \mathcal{E}_i(u, u)$$

For all $u \in \mathcal{F}_i$. The domain of \mathcal{E}_∞ is

$$\mathcal{F}_\infty = \text{clos} \left\{ \bigcup \Phi_i^* \mathcal{F}_i \right\}$$

where the closure is in the $\mathcal{E}_\infty + \|\cdot\|_\infty$ metric.

Theorem 4.1. *If $(\mathcal{E}_0, \mathcal{F}_0)$ is a regular Dirichlet form then the pair $(\mathcal{E}_\infty, \mathcal{F}_\infty)$ is a regular Dirichlet form. Furthermore, if \mathcal{E}_0 is local then \mathcal{E}_∞ is local as well.*

Proof. Linearity, the Markovian property, and positivity follow immediately from the definition. What remains to check is that \mathcal{E}_∞ is closed and regular. The proof that \mathcal{E}_∞ is a closed form follows the same type of argument as used in Theorem 2.1. From that theorem we have that all the \mathcal{E}_i are closed. Using the same criterion, we assume that $u_n \in \bigcup \Phi_i^* \mathcal{F}_i$ is Cauchy in the metric given by $\mathcal{E}_\infty(\cdot, \cdot) + \|\cdot\|_\infty$ and that $\|u_n\|_\infty \rightarrow 0$. Then we need to show that $\mathcal{E}_\infty(u_n) \rightarrow 0$. This will show that $(\mathcal{E}_\infty, \bigcup \Phi_i^* \mathcal{F}_i)$ is a closable quadratic form and the closure will be $(\mathcal{E}_\infty, \mathcal{F}_\infty)$ as defined above. Let $\epsilon > 0$ then there exists $N \geq 0$ such that $\|u_n\|_\infty < \epsilon$ for $n \geq N$. Then there exists M large enough such that $u_n \in \mathcal{F}_M$ for $n \leq M$ and since $\mathcal{E}_M(u_M) < \epsilon$. Such a finite M exists because all of the \mathcal{E}_i are closed and the first M of the u_n form the beginning of a Cauchy sequence in $\mathcal{E}_M(\cdot, \cdot) + \|\cdot\|_M$. This M depends only on ϵ so by taking a sequence $\epsilon_p \rightarrow 0$ there is a subsequence u_{n_p} along which $\mathcal{E}_\infty(u_{n_p}) \rightarrow 0$. Since this was a Cauchy sequence by assumption we have $\mathcal{E}_\infty(u_n) \rightarrow 0$.

The next claim is that \mathcal{E}_∞ is regular. This requires a statement of what the continuous functions on $\lim_{\leftarrow} F_i$ are. The continuous functions on the limit space are those functions that are in the uniform closure of functions of the form $\Phi_i^* f$ for some $i \geq 0$ and $f \in C(F_i)$. This is from the topology given by the inverse limit construction on the limit space. Elements of \mathcal{F}_∞ can be used to approximate elements of $C(F_i)$ by the regularity of \mathcal{E}_i any continuous function on $\lim_{\leftarrow} F_i$ can be approximated by a sequence in \mathcal{F}_∞ by taking a diagonal sequence. Similarly one can take a diagonal sequence of continuous functions approximating elements of \mathcal{F}_∞ .

Finally, we claim that if the \mathcal{E}_i are local then \mathcal{E}_∞ is local as well. Let $f, g \in \mathcal{F}_\infty$ with disjoint compact support. Then f and g are approximable in \mathcal{E}_∞ by f_n and g_n such that for each n f_n and g_n have disjoint compact support. Since \mathcal{E}_∞ is a closed form

$$\mathcal{E}_\infty(f, g) = \lim_{n \rightarrow \infty} \mathcal{E}_\infty(\Phi_n^* f_n, \Phi_n^* g_n) = \lim_{n \rightarrow \infty} \mathcal{E}_n(f_n, g_n) = 0.$$

□

Theorem 4.2. *If Δ_i is the Laplacian generated by \mathcal{E}_i and $\Phi_j : \lim_{\leftarrow} F_i \rightarrow F_j$ the continuous projection from the projective limit construction. Then*

$$\Phi_{i-1}^* \text{Dom}(\Delta_{i-1}) \subset \Phi_i^* \text{Dom}(\Delta_i) \quad \forall i \geq 0.$$

Proof. For a general Dirichlet form $g \in \text{Dom}(\Delta)$ if and only if there exists $f \in L^2$ such that for all $v \in \mathcal{F}$ that

$$\mathcal{E}(g, v) = \langle f, v \rangle_{L^2}.$$

It is sufficient to check that if $u \in \text{Dom}(\Delta_{i-1})$ then $\Phi_i^* u \in \text{Dom}(\Delta_i)$. We shall use Proposition 2.2 to ensure that $\pi_i^* v(\cdot, g) \in \mathcal{F}_{i-1}$ for a.e. $-g$. Let $u \in \text{Dom}(\Delta_{i-1})$ and $v \in \mathcal{F}_i$.

$$\begin{aligned}
\mathcal{E}_i(\phi_i^* u, v) &= \int_{G_i} \mathcal{E}_{i-1}(\pi_i^* \phi_i^* u, \pi_i^* v)(g) \, d\mu_{G_i} \\
&= \int_{G_i} \mathcal{E}_{i-1}(u, \pi_i^* v) \, d\mu_{G_i} \\
&= \int_{G_i} \int_{F_{i-1}} (\Delta_{i-1} u)(\pi_i^* v) \, d\mu_{F_i} \, d\mu_{G_i} \\
&= \int_{F_{i-1} \times G_i} (\Delta_{i-1} u)(\pi_i^* v) \, d\mu_{F_{i-1} \times G_i} \\
&= \int_{F_i} (\phi_i^* u) v \, d\mu_{F_i}
\end{aligned}$$

Thus $\phi_i^*(\Delta_{i-1} u) = \Delta_i \phi_i^* u$. □

Definition 4.2. For $i \geq 1$, given a Borel measurable $f : F_i \rightarrow \mathbb{R}$ define the projections \mathcal{P}_i and $\tilde{\mathcal{P}}_i$ by

$$(\mathcal{P}_i f)(x) = \phi_i^* \left(\int_{G_i} (\pi_i^* f)(x, g) \, d\mu_{G_i} \right)$$

and

$$(\tilde{\mathcal{P}}_i f)(x) = \int_{G_i} (\pi_i^* f)(x, g) \, d\mu_{G_i}.$$

These projection can be defined with domains $C_0(F_i)$, $L^2(F_i)$, or any subsets of these. The domain will be made clear in each context.

The integral in this definition maps a function on $F_{i-1} \times G_i$ to a function on F_{i-1} so that \mathcal{P}_i takes functions on F_i and returns another function on F_i . Note that for $x \in B_i$ that $\mathcal{P}_i f(x) = f(x)$. Because $\pi_i^* f(x, g)$ is constant over all the values of g if $x \in B_i$. However the $\tilde{\mathcal{P}}_i$ can be composed to project down several levels. Let Π_i be the left inverse of Φ_i^* , then the families \mathcal{P}_i and Π_i satisfy the following relation for $f \in L^2(F_i)$

$$\Pi_{i-1} \circ \Phi_i^*(f) = \tilde{\mathcal{P}}_i(f).$$

Lemma 4.1. The generator of \mathcal{E}_∞ denoted Δ_∞ is the weak limit of $\Phi_i^* \Delta_i \Pi_i$ where

$$\text{Dom}(\Phi_i^* \Delta_i \Pi_i) = \Phi_i^* \text{Dom}(\Delta_i).$$

Proof. First Δ_∞ is the unique maximal self-adjoint operator on $L^2(F_\infty)$ such that for all $f \in \text{Dom}(\Delta_\infty) \subset \mathcal{F}_\infty$ and $g \in \mathcal{F}_\infty$ that

$$\langle \Delta_\infty f, g \rangle = \mathcal{E}_\infty(f, g).$$

For $f \in \text{Dom}(\Delta_i)$ and $g \in \mathcal{F}_i$

$$\begin{aligned}
\mathcal{E}_\infty(\Phi_i^* f, \Phi_i^* g) &= \mathcal{E}_i(f, g) \\
&= \langle \Delta_i f, g \rangle_{L^2(F_i)} \\
&= \langle \Phi_i^* \Delta_i f, \Phi_i^* g \rangle_{L^2(F_\infty)}.
\end{aligned}$$

This equality holds for all $g' \in \mathcal{F}_\infty$ because $\Pi_i g' \in \mathcal{F}_i$. Also there exists $f' \in \mathcal{F}_\infty$ such that $\Pi_i f' = f$, namely $\Phi_i^* f$. Hence this equality can be rewritten as

$$\mathcal{E}_\infty(f', g') = \langle \Phi_i^* \Delta_i \Pi_i f', g' \rangle_{L^2(F_\infty)}.$$

Since Π is an orthogonal projection, Δ_i is self-adjoint, and Φ_i^* is an inclusion map, $\Phi_i^* \Delta_i \Pi_i$ is a self-adjoint operator with domain $\Phi_i^* \text{Dom}(\Delta_i)$. Notice that $\Phi_i^* \Delta_i \Pi_i$ possesses the defining property of

Δ_∞ on $\Phi^* \text{Dom}(\Delta_i)$ so $\Phi^* \text{Dom}(\Delta_i) \subset \text{Dom}(\Delta_\infty)$. We now have

$$\begin{aligned} \langle \Delta_\infty f, g \rangle_{L^2(F_\infty)} &= \mathcal{E}_\infty(f, g) \\ &= \lim_{i \rightarrow \infty} \mathcal{E}_i(\Pi_i f, \Pi_i g) \\ &= \lim_{i \rightarrow \infty} \langle \Delta_i \Pi_i f, \Pi_i g \rangle_{L^2(F_i)} \\ &= \lim_{i \rightarrow \infty} \langle \Phi_i^* \Delta_i \Pi_i f, \Phi_i^* \Pi_i g \rangle_{L^2(F_\infty)} \end{aligned}$$

Where $f \in \text{Dom}(\Delta_\infty)$ and $g \in \mathcal{F}_\infty$ where $\Pi_i : \text{Dom}(\Delta_\infty) \mapsto \text{Dom}(\Delta_i)$ which follows from Lemma 4.4. Since $\Phi_i^* \Pi_i g \rightarrow g$ in \mathcal{F}_∞ with i going to infinity we get that $\Delta_\infty = w - \lim_{i \rightarrow \infty} \Phi_i^* \Delta_i \Pi_i$. \square

Definition 4.3. *Set the following notation:*

$$\begin{aligned} \ker(\mathcal{P}_i|_{C(F_i)}) &= \mathcal{C}_i \\ \ker(\mathcal{P}_i|_{L^2(F_i)}) &= \mathcal{L}_i \\ \ker(\mathcal{P}_i|_{\mathcal{F}_i}) &= \mathcal{F}'_i \end{aligned}$$

Lemma 4.2. *Let \mathcal{P}_i be defined on $C_0(F_i)$. Then*

$$C_0(F_i) = \phi_i^*(C_0(F_{i-1})) \oplus \mathcal{C}_i.$$

Moreover, $h \in \mathcal{C}_i$ if and only if $h(x, g)$ satisfies

$$\int_{G_i} (\pi_i^* h)(x, g) d\mu_{G_i} = 0 \quad \forall x \in F_{i-1}.$$

Proof. First note that any characterization of the kernel of \mathcal{P}_i is a statement about the pull back of functions from $C_0(F_i)$ to $C_0(F_{i-1} \times G_i)$. On $F_{i-1} \times G_i$ one can distinguish two closed sets of functions. The first that are constant over G_i and the second those that have mean zero over G_i for every $x \in F_{i-1}$. These two sets of functions are $\psi_i^*(C_0(F_{i-1}))$ and $\pi_i^* \mathcal{C}_i$ respectively and both are subsets of $\pi_i^* C_0(F_i)$ and only have the constantly zero function in common. Since they are both in $\pi_i^* C_0(F_i)$ consider their images in $C_0(F_i)$, these are now the two summands in the statement of the Lemma.

Take $f \in C_0(F_i)$, we want to write f as the sum of an element in $\phi_i^* C_0(F_{i-1})$ and an element in \mathcal{C}_i . Write $f = \mathcal{P}_i(f) + (f - \mathcal{P}_i(f))$. Then $\mathcal{P}_i(f) \in \phi_i^*(C_0(F_{i-1}))$ since \mathcal{P}_i is the projection onto precisely those functions. Then we need to check that $\pi_i^*(f - \mathcal{P}_i(f))$ has mean zero over G_i for every $x \in F_{i-1}$.

$$\begin{aligned} \int_{G_i} \pi_i^*(f - \mathcal{P}_i(f))(x, g) d\mu_{G_i} &= \int_{G_i} \pi_i^*(f) d\mu_{G_i} - \int_{G_i} \pi_i^* \mathcal{P}_i(f) d\mu_{G_i} \\ &= \int_{G_i} \pi_i^*(f) d\mu_{G_i} \\ &\quad - \int_{G_i} \pi_i^* \phi_i^* \int_{G_i} \pi_i^*(f) d\mu_{G_i} d\mu_{G_i} \end{aligned} \tag{2}$$

In (2) the integrand of the outer integral has no g dependence the integral only serves to push it's integrand back into a function on F_{i-1} . Because Figure 1 is a commutative diagram this integration composed with $\pi_i^* \phi_i^*$ is the identity operator. Thus

$$\int_{G_i} \pi_i^*(f - \mathcal{P}_i(f))(x, g) d\mu_{G_i} = \int_{G_i} \pi_i^*(f) d\mu_{G_i} - \int_{G_i} \pi_i^*(f) d\mu_{G_i} = 0.$$

\square

Lemma 4.3. *Let \mathcal{P}_i be defined on $L^2(F_i)$. Then*

$$L^2(F_i) = \phi_i^*(L^2(F_{i-1})) \oplus \mathcal{L}_i.$$

Elements of \mathcal{L}_i have the corresponding property as elements of \mathcal{C}_i for almost every $x \in F_{i-1}$.

Proof. This Lemma follows the same ideas as the previous however the technicalities are reduced because we are working with a Hilbert space. If \mathcal{P}_i is the projection onto $\phi_i^*(L^2(F_{i-1}))$ then the claimed decomposition is just the orthogonal decomposition. By definition \mathcal{P}_i is a projection so it only remains to show that it is only $\phi_i^*(L^2(F_{i-1}))$. Take any element $f \in \phi_i^*(L^2(F_{i-1}))$ and calculate $\mathcal{P}_i(f)$. This is a short calculation which immediately shows that $f = \mathcal{P}_i(f)$ in $L^2(F_i)$. \square

Lemma 4.4. *Let \mathcal{P}_i be defined on \mathcal{F}_i . Then*

$$\mathcal{F}_i = \phi_i^*(\mathcal{F}_{i-1}) \oplus \mathcal{F}'_i.$$

Elements of \mathcal{F}'_i have the corresponding property as elements of \mathcal{L}_i . Moreover the core, $C(F_i) \cap \mathcal{F}_i$ of the Dirichlet form $(\mathcal{E}_i, \mathcal{F}_i)$ has the same decomposition.

Proof. The decomposition of \mathcal{F}_i is a consequence of Lemma 4.3 and the characterization of the elements of $\ker \mathcal{P}_i$ follows similarly to the previous lemmas. The novel claim is the statement concerning the core of \mathcal{F}_i . The core of \mathcal{F}_i that we chose is $C(F_i) \cap \mathcal{F}_i$. That this is a core is a consequence of the regularity result in Theorem 2.1. Lemma 4.2 gives this kind of decomposition of $C(F_i)$ but it still needs to be checked that if $f \in C(F_i) \cap \mathcal{F}_i$ is decomposed according to Lemma 4.2 that the two components are again in $\phi_i^*(\mathcal{F}_{i-1})$ and \mathcal{F}'_i respectively.

Let $f \in C(F_i) \cap \mathcal{F}_i$ and decompose into $f_{i-1} \in C(F_{i-1})$ and $f_c \in \mathcal{C}_i$. We will show that $f_{i-1} \in \phi_i^*(\mathcal{F}_{i-1})$ and $f_c \in \mathcal{F}'_i$. Because $\phi_i^*(\mathcal{F}_{i-1})$ is closed in \mathcal{F}_i and \mathcal{E}_{i-1} is assumed to be regular it is enough to check elements of $\mathfrak{F}_i \cap C(F_i)$ for these properties. Let $f \in \mathfrak{F}_i \cap C(F_i)$ then $(\pi_i^* f)(x, g) = \sum_{j=1}^n f_j(x) h_j(g)$ where the f_j and h_j are all continuous. The action of \mathcal{P}_i on f is $\mathcal{P}_i f = \phi_i^*(\sum_{j=1}^n f_j(x) \int_{G_i} h_j(g) d\mu_{G_i}(g)) = f_{i-1}$. Which is in $\phi_i^*(\mathcal{F}_{i-1})$ by the definition of \mathfrak{F}_i . Set $f_c = f - f_{i-1}$, this is in $\mathcal{F}'_i \cap C(F_i) \subset \mathcal{C}_i$. \square

Definition 4.4. *Let $\mathfrak{D}'_0 = \Phi_0^* \text{Dom}(\Delta_0)$. The inductively define \mathfrak{D}'_i by:*

$$\mathfrak{D}'_i = \Phi_i^* \text{Dom}(\Delta_i) \cap \mathfrak{D}'_{i-1}{}^\perp.$$

Where the orthogonal complement is taken in $L^2(F_i)$. This implies that $\Phi_i^ \text{Dom}(\Delta_i) = \oplus_{j=0}^i \mathfrak{D}'_j$.*

Theorem 4.3.

$$\begin{aligned} L^2(F_\infty, \mu_\infty) &= \text{clos}_{L^2(F_\infty, \mu_\infty)} (\Phi_0^* L^2(F_0, \mu_{F_0}) \oplus (\oplus_{i=1}^\infty \Phi_i^* \mathcal{L}_i)) \\ C(F_\infty) &= \text{clos}_{\text{unif}} (\Phi_0^* C(F_0) \oplus (\oplus_{i=1}^\infty \Phi_i^* \mathcal{C}_i)) \\ \mathcal{F}_\infty &= \text{clos}_{\mathcal{F}_\infty} (\Phi_0^* \mathcal{F}_0 \oplus (\oplus_{i=1}^\infty \Phi_i^* \mathcal{F}'_i)) \end{aligned}$$

Proof. By definition $L^2(F_\infty, \mu_{F_\infty})$ is the completion of $\bigcup_{i=0}^\infty \Phi_i^* L^2(F_i, \mu_{F_i})$ what is new is the direct sum decomposition. Let $f \in L^2(F_1, \mu_1)$ then notice that $f = (f - \mathcal{P}_1 f) + \phi_1^*(\tilde{\mathcal{P}}_1 f) \in \mathcal{L}_1 \oplus \phi_1^* L^2(F_0)$. In general for $f \in L^2(F_2)$ we would have

$$f = (f - \mathcal{P}_2 f) + \phi_2^*(\tilde{\mathcal{P}}_2 f - \mathcal{P}_1 \tilde{\mathcal{P}}_2 f) + \phi_2^* \phi_1^*(\tilde{\mathcal{P}}_1 \tilde{\mathcal{P}}_2 f) \in \mathcal{L}_2 \oplus \phi_2^* \mathcal{L}_1 \oplus \phi_2^* \phi_1^* L^2(F_0).$$

Continuing by this method we have the direct sum expansion for $L^2(F_i)$ for any $i \geq 1$. The $L^2(F_\infty)$ limits of these expansions must then be all of $L^2(F_\infty)$. The same argument works for $C(F_\infty)$ and \mathcal{F}_∞ . \square

The domain of Δ_∞ can be decomposed into the direct sum of \mathfrak{D}'_i , or as $\text{Dom}(\Delta_\infty) \cap \mathcal{L}_i$ or as $\text{Dom}(\Delta_\infty) \cap \mathcal{F}'_i$.

Lemma 4.5. *The three direct sum decompositions of $\text{Dom}(\Delta_\infty)$ agree.*

Proof. Because $\mathcal{F}_\infty \subset L^2(F_\infty)$ we know that $\mathcal{F}'_i \subset \mathcal{L}_i$. This together with the fact that $\text{Dom}(\Delta_\infty) \subset \mathcal{F}_\infty$ implies that $\text{Dom}(\Delta_\infty) \cap \mathcal{L}_i = \text{Dom}(\Delta_\infty) \cap \mathcal{F}'_i$. What remains to be shown is that when the closure of the direct sum is taken that all of $\text{Dom}(\Delta_\infty)$ is obtained as limits of elements of $\text{Dom}(\Delta_\infty) \cap \mathcal{F}'_i$. Suppose there exists a $g \in \text{Dom}(\Delta_\infty)$ such that for any sequence $g_n \in \mathcal{F}_0 \oplus [\oplus_{i=0}^n (\text{Dom}(\Delta_\infty) \cap \mathcal{F}'_i)]$ there exists $\epsilon > 0$ such that $\|g - g_n\|_{L^2} + \|\Delta_\infty(g - g_n)\|_{L^2} > \epsilon$. That is $g \in \text{Dom}(\Delta_\infty)$ but not in the

closure of $\oplus_{i=0}^{\infty} (Dom(\Delta_{\infty}) \cap \mathcal{F}'_i)$. Then choose g_n such that $g_n - g_{n-1} \in \mathcal{F}'_n$ and $g_n \rightarrow g$ in \mathcal{F}_{∞} and in $L^2(F_{\infty})$. Hence $\|\Delta_{\infty}(g - g_n)\|_{L^2} > \epsilon$. So

$$\begin{aligned} \epsilon^2 < \|\Delta_{\infty}(g - g_n)\|_{L^2}^2 &\leq \sqrt{\mathcal{E}_{\infty}(g - g_n)} \|\Delta_{\infty}(g - g_n)\| \\ &\quad + \sqrt{\mathcal{E}_{\infty}(\Delta_{\infty}(g - g_n))} \|g - g_n\| \\ \frac{\|g - g_n\| - \sqrt{\mathcal{E}_{\infty}(\Delta_{\infty}(g - g_n))} \|g - g_n\|}{\|\Delta_{\infty}(g - g_n)\|} &\leq \sqrt{\mathcal{E}_{\infty}(g - g_n)}. \end{aligned}$$

As $n \rightarrow \infty$ the right hand side goes to zero by the choice of g_n . However since $\|\Delta_{\infty}(g - g_n)\|_{L^2} > \epsilon$ the left hand side goes to one. So the g did not exist. \square

5. MAIN RESULTS

Theorem 5.1. *The spectrum of Δ_{∞} is given by:*

$$\sigma(\Delta_{\infty}) = \overline{\bigcup_{i=0}^{\infty} \sigma(\Delta_i)} = \overline{\bigcup_{i=0}^{\infty} \sigma(\Delta_{\infty}|_{\mathfrak{D}'_i})}$$

Proof. Let $z \in \sigma(\Delta_n)$. Then by Lemma 4.1 $(\Delta_n - z)$ is not invertible on $Dom(\Delta_n)$. Since $(\Delta_{\infty} - z)$ agrees with $(\Delta_n - z)$ on $\Phi_n^* Dom(\Delta_n) \subset Dom(\Delta_{\infty})$ we have that $(\Delta_{\infty} - z)$ is not invertible. Hence $\sigma(\Delta_n) \subset \sigma(\Delta_{\infty})$ for all $n \geq 0$. This implies that $\sigma(\Delta_n) = \sigma(\Delta_{\infty}|_{\mathfrak{D}_n})$ where $\mathfrak{D}_n = \oplus_{i=0}^n \mathfrak{D}'_i$ and $\sigma(\Delta_{\infty}|_{\mathfrak{D}_n}) = \bigcup_{i=0}^n \sigma(\Delta_{\infty}|_{\mathfrak{D}'_i})$.

Suppose that $z \in \sigma(\Delta_{\infty})$ and $z \notin \overline{\bigcup_{i=0}^{\infty} \sigma(\Delta_i)}$. Define $B_z : L^2(F_{\infty}) \rightarrow Dom(\Delta_{\infty})$ by

$$B_z = w - \lim_{i \rightarrow \infty} \Phi_i^*(\Delta_i - z)^{-1} \Pi_i.$$

We claim that B_z is the inverse of $(\Delta_{\infty} - z)$ contradicting the assumption that $z \in \sigma(\Delta_{\infty})$. Recall that Π_i is a bounded linear and hence continuous operator. Let $f \in \bigcup_{i=0}^{\infty} \Phi_i^* Dom(\Delta_i)$, then

$$\begin{aligned} (3) \quad B_z(\Delta_{\infty} - z)f &= \lim_{n \rightarrow \infty} \Phi_n^*(\Delta_n - z)^{-1} \Pi_n \lim_{m \rightarrow \infty} \Phi_m^*(\Delta_m - z) \Pi_m f \\ &= \lim_{n \rightarrow \infty} \Phi_n^*(\Delta_n - z)^{-1} \Pi_n \Phi_M^*(\Delta_M - z) \Pi_M f \\ &= \Phi_M^*(\Delta_M - z)^{-1} (\Delta_M - z) \Pi_M f \\ &= f. \end{aligned}$$

For large enough m $\lim_{m \rightarrow \infty} \Phi_m^*(\Delta_m - z) \Pi_m f$ stabilizes to $\Phi_M^*(\Delta_M - z) \Pi_M f$ then as n grows (3) will also stabilize for $n \geq M$. Then since Δ_{∞} is a closed operator the claim extends to $Dom(\Delta_{\infty})$. Finally by the decompositions in Lemmas 4.3 and 4.4 the last limit equals f . The same calculation can be used to show that $(\Delta_{\infty} - z)B_z = Id$. Thus there exists no $z \in \sigma(\Delta_{\infty})$ but not in $\overline{\bigcup_{i=0}^{\infty} \sigma(\Delta_i)}$. \square

In the standard theory of self-adjoint operators lie the spectral resolutions of self-adjoint operators [19]. These spectral resolutions are orthogonal projection valued measures over \mathbb{R} supported on the spectrum of the operator they are representing. For Δ_{∞} let E_{λ} be the spectral resolution. Then

$$\Delta_{\infty} f = \int_{\sigma(\Delta_{\infty})} \lambda dE_{\lambda} f.$$

Note that for each $\lambda \in \mathbb{R}$, $E_{\lambda} : L^2(F_{\infty}) \rightarrow Dom(\Delta_{\infty})$ where for $f \notin Dom(\Delta_{\infty})$ the integral fails to converge. We also have the orthogonal projections \mathcal{P}_i out of $Dom(\Delta_{\infty})$.

From the previous discussion the following statement follows immediately.

Theorem 5.2. *Let E_{λ} be a spectral projection operator for Δ_{∞} . Then for all $\lambda \in \mathbb{R}$ and $i \in \mathbb{N}$*

$$\mathfrak{D}'_i \cap E_{\lambda}(Dom(\Delta_{\infty})) = E_{\lambda} \mathfrak{D}'_i.$$

Similar statements could be made for $L^2(F_{\infty})$, \mathcal{F}_{∞} , however we have not developed the notation for these spaces corresponding to the \mathfrak{D}'_i notation.

Theorem 5.3. *Suppose that \mathcal{E}_0 is a local regular Dirichlet form. If G_i have bounded, finite cardinality, F_0 is compact, $F_i \setminus B_i$ consists of disjoint open sets whose diameters are monotonically decreasing to zero in i and the cardinality of B_i is finite and increasing then $\sigma(\Delta_i)$ are all discrete and $\sigma(\Delta_\infty) = \bigcup_{i=0}^\infty \sigma(\Delta_n)$.*

Proof. By the finiteness of the G_i we have that $F_\infty = F_0 \times K / \sim$ where K is a Cantor set and the equivalence relation \sim is the union of all the \sim_i . Which implies that F_∞ is compact. If it can be shown that $\min \mathfrak{D}_i$ increases without bound then the union will be nowhere dense and no new eigenvalues will be generated from the closure.

By the assumptions on the cardinality of B_i and that $F_i \setminus B_i$ each eigenfunction of Δ_i is decomposable into eigenfunctions of Δ_{i-1} on each connected component of $F_i \setminus B_i$ which is isomorphic to an open subset of F_{i-1} . Let $f \in \mathfrak{D}_i$ with eigenvalue $\lambda_i = \min \sigma(\Delta_i|_{\mathfrak{D}_i})$. Then f restricted to a single component of $F_i \setminus B_i$ is locally an eigenfunction of Δ_{i-1} with eigenfunction λ_i and Dirichlet boundary conditions at B_i . If Δ_{i-1} has a fundamental solution which is strictly positive then λ_i must be larger than λ_{i-1} by the assumption on the sizes of the open components of $F_i \setminus B_i$. \square

6. EXAMPLES

6.1. The Laakso fractal and the heat kernel estimates. Laakso spaces were initially introduced in [18] using the Cartesian product of a unit interval and a number of Cantor sets. In [23, 21] it was shown that they could also be constructed using the projective limit construction presented originally in [10] and reiterated above. Take $F_0 = [0, 1]$, the unit interval. Let $G_i = G = \{0, 1\}$. Choose a sequence $\{j_l\}_{l=1}^\infty$ where $j_l \in \{j, j+1\}$ for some fixed integer greater than one. Define

$$d_N = \prod_{j=1}^N j_i \quad L_N = \left\{ \frac{i}{d_N} \right\}_{i=1}^{d_N-1}.$$

Then set $B_n = \phi_{n,0}^{-1}(L_n \setminus L_{n-1})$. The sets L_N describe the location of what the quotient maps π_i collapse and d_N the separation of the new identifications from any old identifications. A Laakso space will be denoted by L .

If \mathcal{E}_0 is taken to be the simplest Dirichlet form on the unit interval, namely $\mathcal{E}_0(u, v) = \int_0^1 \frac{du}{dx} \frac{dv}{dx} dx$ with the Sobolev space $H^{1,2}([0, 1])$ as \mathcal{F}_0 , then there is a limiting Dirichlet form, \mathcal{E}_∞ , on L which has a generator Δ_∞ . The analysis of Δ_∞ 's spectrum is the topic of [21] and several chapters in [23]. Using the arguments involved in the proofs of Theorems 5.1 and 5.3 the following results are known.

Theorem 6.1 ([21]). *Let L be a Laakso space with sequence $\{j_i\}$. The spectrum of Δ_∞ on this Laakso space is*

$$\sigma(A) = \bigcup_{n=0}^\infty \bigcup_{k=1}^\infty \{k^2 \pi^2 d_n^2\} \cup \bigcup_{n=2}^\infty \bigcup_{k=1}^\infty \{k^2 \pi^2 4d_n^2\} \cup \bigcup_{n=1}^\infty \bigcup_{k=0}^\infty \{(2k+1)^2 \pi^2 4d_n^2\}.$$

6.2. Sierpinski Pâte à Choux. The name of this example was suggested by Jean Bellissard who commented that such a space would evoke the memory of puff pastry in the reader. Denote by SG the standard Sierpinski gasket constructed as the limit of the iterated function system $T_l(x) = \frac{1}{2}(x - q_l) + q_l$ for $l = 0, 1, 2$ where $q_0 = (0, 0)$, $q_1 = (1, 0)$, and $q_2 = (\frac{1}{2}, \sqrt{3}/2)$. Let $F_0 = SG$, $G_i = G = \{0, 1\}$ and $B_i = V_i \setminus V_{i-1}$. We use the convention that V_n consists the images of the points q_l under all the n -fold compositions of the contractions T_j .

Lemma 6.1. *The limit space F_∞ is an infinitely ramified fractal with Hausdorff dimension $d_h = 1 + d_H(SG) = \frac{\log(6)}{\log(2)}$ with respect to the geodesic metric.*

Proof. The cell structure on F_∞ induced by the cell structures on SG and on the Cantor set have boundaries that are themselves Cantor sets. Hence F_∞ is infinitely ramified. Since $\iota^{-1}(F_\infty) \subset SG \times K$ the Hausdorff dimension is at most $\frac{\log(6)}{\log(2)}$. By the same argument as in [24] it is at least $\frac{\log(6)}{\log(2)}$. \square

In light of Theorem 5.3 it would be possible to write out explicitly the spectrum on F_∞ as we did with the Laakso spaces. In particular, it is possible but somewhat involved to write the spectrum in a closed form. The reader can find solution to a similar problem in [26]. We note that, in the limit, the Sierpinski Pâte à Choux is not a Sierpinski fractafold, but the approximations F_i are fractafolds. However, despite the fact that these fractafolds are very complicated, the spectrum of the Laplacian on F_i can be found inductively using methods presented in this paper. In particular, the spectrum of each Laplacian Δ_i is a union of the spectrum of a large collection of disjoint fractafolds (with Dirichlet boundary conditions). These fractafolds are rescaled copies of two kinds of finite fractafolds, and therefore the spectrum can be found using the methods of [25], and the standard rescaling by 5^n . This is very similar to how the spectrum is found in the case of the Laakso spaces, described above. Finally, we can comment that the Laakso spaces are built using intervals, which are one-dimensional analogs of the Sierpinski gasket. Therefore, in a sense, the Sierpinski Pâte à Choux is a direct analog of the Laakso spaces. Combining the approaches of [15, 23, 21, 25, 26] one can study all the eigenfunctions and eigenprojections, which will be subject of subsequent work.

6.3. Connected fractal spaces isospectral to the fractal strings of Lapidus and van Franken-huijsen. Fractal strings are given a comprehensive treatment in [20], in particular in relation to spectral zeta functions, and we will only give a brief description here. We show that our construction can yield connected fractal spaces with Laplacians isospectral to the standard Laplacians on fractal strings. This implies, in particular, that there are symmetric irreducible diffusion processes whose generators are Laplacians with prescribed spectrum, as in the theory of fractal strings developed in [20].

A fractal string is an open subset of \mathbb{R} , usually assumed to be a bounded subset, or at least that the lengths of intervals are bounded and tend to zero. Therefore it is a union of countably many finite intervals of lengths l_i . We will suppose that the intervals are indexed so that the lengths form a non-increasing sequence. By indexing the fractal string with l_i and m_i , lengths and multiplicities we can assume that l_i is strictly decreasing. The Laplacian that we consider on a I is the usual Laplacian on an interval with Dirichlet boundary conditions on all the intervals. The eigenvalues of this Laplacian are all of the form

$$\lambda_{i,k} = \frac{\pi^2 k^2}{l_i^2}$$

with multiplicity m_i . What choices of F_i , B_i , and G_i can be made to create a connected fractal with the same spectrum as a given fractal string? As the desire is to “stitch” the disjoint intervals together there is no unique canonical method.

To begin with, we let $F_0 = [0, l_1]$, $B_1 = \{0, 1\}$ and $G_1 = \{1, 2, \dots, m_1\}$. Then F_1 will be m_1 copies of the unit interval with left end points identified and right end points identified. A particular implication of this step is that $F_0 = F_1$ if and only if $m_1 = 1$. We impose zero boundary conditions at the endpoints, and therefore the spectrum of the Laplacian on F_1 is the spectrum on $F_0 = [0, l_1]$ repeated, in the sense of multiplicity, m_1 times. For the next step $G_2 = \{1, 2, \dots, m_2 + 1\}$, and we choose

$$B_2 = ([0, 1] \times ([1 - l_2, 1] \times \{1\} \cup [0, 1] \times (G_2 \setminus \{1\}))) / \sim_1$$

where \sim_1 is the equivalence relation on $F_0 \times G_1$ determined by B_1 . This implies that the spectrum on F_2 is the union of the spectrum on F_1 and the spectrum on $[0, l_2]$ repeated, in the sense of multiplicity, m_2 times. For $i = 3$ we take

$$B_3 = [1 - l_3, 1] \times \{(1, 1)\} \cup [0, 1] \times (G_1 \times G_2 \setminus \{(1, 1)\}).$$

Then B_n is constructed inductively in the same manner. This construction is in a sense a non-self-similar version of the nested fractal construction. It is also somewhat similar to construction of some of the so called diamond fractals, see [1, 22]

In this setting one cannot employ Theorem 5.3 directly. However it is easy to replace the condition on the cardinality of B_i with our specific choice of B_i , and the same result can be shown. Namely,

the spectrum of the Laplacian Δ is given by the union of the spectra of Δ_n and that \mathfrak{D}_i are functions with eigenvalues $\lambda_{i,k}$ with multiplicity m_i .

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